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Analysis of L-integral and theory of the derivative stress field in plane elasticity

Y.Z. Chen

Division of Engineering Mechanics, Jiangsu University of Science and Technology, Zhenjiang, Jiangsu 212013, PR China

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Abstract

In this paper, analysis of the L-integral in plane elasticity is present. An infinite plate with any number of inclusions and cracks and with any remote tractions is assumed in analysis. Arbitrary forces are applied on the cracks, inclusions or at a point of the infinite medium. To study the problem, the concept of the derivative stress field is introduced, which is derived from a physical stress field. The mutual work difference integral (MWDI) is also introduced, which is defined as a difference of mutual works done by each other from the physical stress field and the derivative field. It is proved that the $L(\mathrm{CR})$ (L-integral on a large circle) is equal to a particular MWDI. General expression for the $L(\mathrm{CR})$ is obtained. For a given stress field, the variation of the $L(\mathrm{CR})$ is studied when the coordinates have a translation or rotation. It is found that the $L(\mathrm{CR})$ is an invariant with respect to the rotation of coordinates, and it has a variation when the coordinates have a translation.

Keywords: Path independent integral; Fracture mechanics; Plane elasticity

1. Introduction

Knowles and Sternberg (1972) established a total four conservation laws, or path independent integrals in two-dimensional deformation field. These integrals are J_1-, J_2-, L- and M-integral. Budiansky and Rice (1973) have interpreted J_1-, J_2-, L- and M-integral as being the energy release rate when a cavity is translated, is rotated, and is expanded uniformly. These integrals have a general property that the values of the mentioned integrals along a closed path do not depend on the path adopted, provided there is no singularity between two integration paths. Naturally, if the closed path encloses some singularity or cavity, these integrals must not vanish. At the vicinity of the crack tip, the J_i-integral ($i = 1, 2$) has some relation with the stress intensity factors. Also, for the notch case the J_1-integral is an averaged measure of the strain at the notch tip (Rice, 1968).

E-mail address: chens@ujs.edu.cn (Y.Z. Chen).

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Investigation of these integrals in the crack problems received particular interest. In some particular cases, some path independent integrals can be evaluated in a closed form (Rice, 1968; Fruend, 1978; Cherepanov, 1979; Kanninen and Popelar, 1985). For the M-integral, some studies were carried out (Chen, 1986; Suo, 2000; Chen, 2001). For a single crack in an infinite plate with the remote loading σ_x^∞, σ_y^∞ and σ_{xy}^∞, the L-integral was evaluated (Herrmann and Herrmann, 1981).

Recently, the M-integral for two-dimensional solids with interacting microcracks is studied (Chen, 2001). The study is limited to some simple cases where the infinite plate with traction free crack is applied by remote stresses. Clearly, in some complicated cases people were unaware of behaviors of the path independent integrals. For example, in the case that: (a) the infinite plate is subject to the remote tractions, and (b) it contains cracks, cavities and rigid inclusions with some loading on them, the L-integral on a large circle needs to be investigated.

In this paper, general properties of the L-integral on a large circle are analyzed. It is assumed that an infinite plate contains any number of inclusions and cracks. Arbitrary forces are applied on the cracks, inclusions or at a point of the medium. The applied remote stresses are denoted by σ_x^∞, σ_y^∞ and σ_{xy}^∞ (Fig. 1). To study the problem, the mutual work difference integral (MWDI) is introduced, which is defined by the difference of works done by each other from two stress fields on a large circle (Bueckner, 1973; Chen, 1985; Chen and Lee, 2002). The concept of the derivative stress field is introduced. The derivative stress field is a real elasticity solution, which is derived after some manipulation for the physical stress field. It is found that the L-integral on a large circle is equal to an MWDI from the physical stress field and a derivative stress field. Finally, the L-integral on a large circle depends on the following factors: (a) the remote tractions, (b) the resultant forces applied on the defects, (c) the rigid translation term in the complex potentials. The relation between the L-integral and stress intensity factors is also addressed.

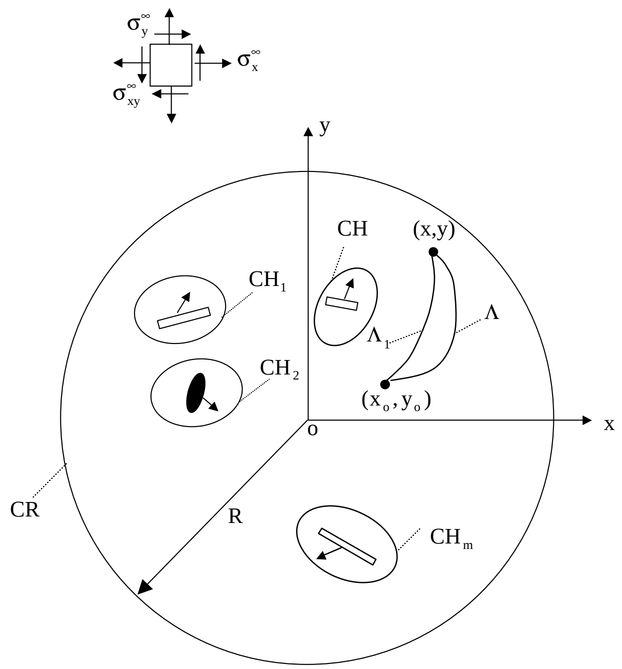

Fig. 1. An infinite plate containing cracks, holes and inclusions.

2. Evaluation of a mutual work difference integral (MWDI) for two physical stress fields

The following analysis depends on the complex variable function method in plane elasticity (Muskhelishvili, 1953). In the method, the stresses $(\sigma_x, \sigma_y, \sigma_{xy})$, the resultant forces (X, Y) and the displacements (u, v) are expressed in terms of two complex potentials $\phi(z)$ and $\psi(z)$ such that

$$\sigma_x + \sigma_y = 4\,\mathrm{Re}\,\phi'(z)$$

$$\sigma_y - \sigma_x + 2\mathrm{i}\sigma_{xy} = 2[\bar{z}\phi''(z) + \psi'(z)] \quad (1)$$

$$f = -Y + \mathrm{i}X = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \quad (2)$$

$$2G(u + \mathrm{i}v) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} \quad (3)$$

where G is the shear modulus of elasticity, $\kappa = (3 - v)/(1 + v)$ is for the plane stress problem, $\kappa = 3 - 4v$ is for the plane strain problem, and v is the Poisson ratio.

In this paper, stress analysis for an infinite plate containing many cracks, inclusions and holes is considered (Fig. 1). The remote stresses are denoted by σ_x^∞, σ_x^∞ and σ_{xy}^∞. Some forces may be applied on the cracks, inclusions and holes.

The physical stress field is defined such that:

(a) In the field, the complex potentials $\phi(z)$ and $\psi(z)$ can be expressed in the form

$$\phi_{(\alpha)}(z) = \phi(z) = A_1 z + A_2 \log z + a_0 + \sum_{k=1}^{\infty} \frac{a_k}{z^k} \quad (4)$$

$$\psi_{(\alpha)}(z) = \psi(z) = B_1 z + B_2 \log z + b_0 + \sum_{k=1}^{\infty} \frac{b_k}{z^k} \quad (5)$$

where A_1, a_0, B_1, b_0 and a_k, b_k $(k = 1, 2, \ldots)$ are some constants. The mentioned stress field is named the α-field hereafter.

(b) In Eqs. (4) and (5), there has a relation between A_2 and B_2,

$$B_2 = -\kappa\overline{A}_2 \quad (6)$$

It is well known that Eq. (6) is a necessary condition for the single-valuedness of displacements (Muskhelishvili, 1953).

In addition, we have (Muskhelishvili, 1953)

$$A_1 = \frac{\sigma_x^\infty + \sigma_y^\infty}{4}, \quad A_2 = -\frac{F_x + \mathrm{i}F_y}{2\pi(\kappa + 1)} \quad (7)$$

$$B_1 = \frac{\sigma_y^\infty - \sigma_x^\infty}{2} + \mathrm{i}\sigma_{xy}^\infty, \quad B_2 = -\kappa\overline{A}_2 = \frac{\kappa(F_x - \mathrm{i}F_y)}{2\pi(\kappa + 1)} \quad (8)$$

In Eqs. (7) and (8), σ_x^∞, σ_y^∞ and σ_{xy}^∞ are the remote stresses, and F_x and F_y are the resultant forces applied on the finite region of infinite plate. In Eqs. (4) and (5), the coefficients a_k and b_k $(k = 1, 2, \ldots)$ will be determined from a concrete solution. Meantime, a_0 and b_0 represent a rigid motion and have no relation with the stress and strain. In the following, it will be found that the two values (a_0 and b_0) are involved in the MWDI for two physical stress fields.

Secondly, the another physical stress field with the name of β-field is introduced, which is defined by

$$\phi_{(\beta)}(z) = E_1 z + E_2 \log z + e_0 + \sum_{k=1}^{\infty} \frac{e_k}{z^k} \tag{9}$$

$$\psi_{(\beta)}(z) = F_1 z + F_2 \log z + f_0 + \sum_{k=1}^{\infty} \frac{f_k}{z^k} \tag{10}$$

As before, E_1, e_0, F_1, f_0 and e_k, f_k $(k = 1, 2, \ldots)$ are arbitrary constants, and E_2 and F_2 satisfy the following condition:

$$F_2 = -\kappa \overline{E}_2 \tag{11}$$

From the mentioned α-field and β-field, an MWDI for two physical stress fields is defined as (Bueckner, 1973; Chen, 1985; Chen and Lee, 2002)

$$D(\mathrm{CR}) = \frac{1}{2} \oint_{(\mathrm{CR})} (u_{i(\alpha)}\sigma_{ij(\beta)} - u_{i(\beta)}\sigma_{ij(\alpha)}) n_j \, \mathrm{d}s \tag{12}$$

where "CR" is a large circle with a radius "R" (Fig. 1), $u_{i(\alpha)}$ ($u_{i(\beta)}$) are the displacements, $\sigma_{ij(\alpha)}$ ($\sigma_{ij(\beta)}$) are the stresses, for the α-field (β-field), respectively, n_j denotes the direction cosine.

In the analysis, the following notations are used:

$$\begin{aligned} &U_{(\alpha)} = u_{(\alpha)} + \mathrm{i}v_{(\alpha)}, \quad U_{(\beta)} = u_{(\beta)} + \mathrm{i}v_{(\beta)} \\ &f_{(\alpha)} = -Y_{(\alpha)} + \mathrm{i}X_{(\alpha)} \quad (\text{or } -\mathrm{i}f_{(\alpha)} = X_{(\alpha)} + \mathrm{i}Y_{(\alpha)}) \end{aligned} \tag{13}$$

$$f_{(\beta)} = -Y_{(\beta)} + \mathrm{i}X_{(\beta)} \quad (\text{or } -\mathrm{i}f_{(\beta)} = X_{(\beta)} + \mathrm{i}Y_{(\beta)}) \tag{14}$$

Using the introduced notations, the first term in Eq. (12) can be rewritten in the form

$$\begin{aligned} u_{i(\alpha)}\sigma_{ij(\beta)} n_j \, \mathrm{d}s &= u_{(\alpha)} \, \mathrm{d}X_{(\beta)} + v_{(\alpha)} \, \mathrm{d}Y_{(\beta)} = \mathrm{Re}[(u_{(\alpha)} - \mathrm{i}v_{(\alpha)}) \, \mathrm{d}(X_{(\beta)} + \mathrm{i}Y_{(\beta)})] = \mathrm{Re}[(-\mathrm{i})(\overline{U}_{(\alpha)} \, \mathrm{d}f_{(\beta)})] \\ &= \mathrm{Im}[\overline{U}_{(\alpha)} \, \mathrm{d}f_{(\beta)}] \end{aligned} \tag{15}$$

Similar derivation is done for the second term in Eq. (12). Finally, Eq. (12) can be written as

$$D(\mathrm{CR}) = \frac{1}{2} \mathrm{Im} \oint_{(\mathrm{CR})} [\overline{U}_{(\alpha)} \, \mathrm{d}f_{(\beta)} - \overline{U}_{(\beta)} \, \mathrm{d}f_{(\alpha)}] \tag{16}$$

Clearly, on the large circle CR with a radius "R" the following properties hold (Fig. 1):

$$\bar{z} = \frac{R^2}{z}, \quad \bar{z}z = R^2, \quad \mathrm{d}\bar{z} = -\frac{R^2}{z^2} \, \mathrm{d}z \qquad (\text{for } z \text{ on the large circle CR}) \tag{17}$$

From Eqs. (4), (5), (9) and (10) we can get the following expressions:

$$\begin{aligned} 2G\overline{U}_{(\alpha)} &= 2G(u_{(\alpha)} - \mathrm{i}v_{(\alpha)}) = \kappa \overline{\phi_{(\alpha)}(z)} - \bar{z}\phi'_{(\alpha)}(z) - \psi_{(\alpha)}(z) \\ &= \kappa \left[\overline{A}_1 \frac{R^2}{z} + 2\overline{A}_2 \log R + \bar{a}_0 + \sum_{k=1}^{\infty} \bar{a}_k \left(\frac{z}{R^2} \right)^k \right] - \frac{R^2}{z} \left[A_1 + \frac{A_2}{z} - \sum_{k=1}^{\infty} \frac{k a_k}{z^{k+1}} \right] - \left[B_1 z + b_0 + \sum_{k=1}^{\infty} \frac{b_k}{z^k} \right] \end{aligned} \tag{18}$$

$$\mathrm{d}f_{(\beta)} = \mathrm{d}[\phi_{(\beta)}(z) + z\overline{\phi'_{(\beta)}(z)} + \overline{\psi_{(\beta)}(z)}] = \left\{ \left[E_1 + E_2 \frac{1}{z} - \sum_{k=1}^{\infty} \frac{k e_k}{z^{k+1}} \right] + \left[\overline{E}_1 + 2\overline{E}_2 \frac{z}{R^2} - \sum_{k=1}^{\infty} k(k+2)\bar{e}_k \left(\frac{z}{R^2} \right)^{k+1} \right] + \left[-\overline{F}_1 \frac{R^2}{z^2} - \overline{F}_2 \frac{1}{z} + \sum_{k=1}^{\infty} \frac{k \bar{f}_k}{R^2} \left(\frac{z}{R^2} \right)^{k-1} \right] \right\} \mathrm{d}z \tag{19}$$

In addition, similar expressions can be achieved for $2G\overline{U}_{(\beta)}$ and $\mathrm{d}f_{(\alpha)}$. After taking the following steps:

(a) Substituting Eqs. (18) and (19) and relevant expressions for $2G\overline{U}_{(\beta)}$ and $\mathrm{d}f_{(\alpha)}$ into Eq. (16).
(b) Using the residue theory in the complex variable function.

The final result is obtained as follows:

$$D(\mathrm{CR}) = \frac{1}{2} \oint_{(\mathrm{CR})} (u_{i(\alpha)}\sigma_{ij(\beta)} - u_{i(\beta)}\sigma_{ij(\alpha)}) n_j \, \mathrm{d}s = \frac{1}{2} \mathrm{Im} \oint_{(\mathrm{CR})} [\overline{U}_{(\alpha)} \, \mathrm{d}f_{(\beta)} - \overline{U}_{(\beta)} \, \mathrm{d}f_{(\alpha)}] = \frac{\pi(\kappa+1)}{2G} \mathrm{Re}[A_1 f_1 + A_2 f_0 - a_0 F_2 - a_1 F_1 + B_1 e_1 + B_2 e_0 - b_0 E_2 - b_1 E_1] \tag{20}$$

It should be noted that a term $-(\kappa\overline{A}_2 + B_2)\log z$ exists in the process of deriving the term $2G\overline{U}_{(\alpha)}$. Because of Eq. (6), this term becomes vanishing. It is easy to see that if Eqs. (6) and (11) do not satisfy, the integral $D(\mathrm{CR})$ will be divergent.

The result shown by Eq. (20) plays an important role in the present study. It will be shown later the J_1-, J_2- , L- and M- integral on a large circle can be reduced to some particular MWDI (Chen, 1985; Chen and Lee, 2002).

3. The derivative stress field in plane elasticity

In the present study, the actual stress field in an infinite medium under the action of remote loads and forces applied on finite portion is called the original stress field, or in turn is called the α-field (Fig. 1). In order to simplify the written form, the subscript "(α)" is generally omitted, for example, $\phi_{(\alpha)}(z)$, $\psi_{(\alpha)}(z)$, $u_{i(\alpha)}$, $\sigma_{ij(\alpha)}$, $u_{(\alpha)}$, $\sigma_{x(\alpha)}$ are rewritten as $\phi(z)$, $\psi(z)$, u_i, σ_{ij}, u, σ_x.

The derivative stress field is defined such that:

(a) The derivative stress field is generally derived from the original field (the α-field).
(b) The relevant complex potentials in the derivative field should satisfy the single-valuedness condition of displacements, or the equality shown by Eqs. (6) or (11).

The following are two examples. In the first example, the complex potentials for β-field are defined as

$$\phi_{(\beta)}(z) = c_1\phi(z) = c_1 \left\{ A_1 z + A_2 \log z + a_0 + \sum_{k=1}^{\infty} \frac{a_k}{z^k} \right\} \tag{21}$$

$$\psi_{(\beta)}(z) = c_2\psi(z) = c_2 \left\{ B_1 z + B_2 \log z + b_0 + \sum_{k=1}^{\infty} \frac{b_k}{z^k} \right\} \quad (c_1 \neq c_2) \tag{22}$$

where c_1 and c_2 are two real constants, and $c_1 \neq c_2$. Comparing Eqs. (9) and (10) with Eqs. (21) and (22), it follows

$$E_2 = c_1 A_2, \quad F_2 = c_2 B_2 \tag{23}$$

In this case, since $B_2 = -\kappa \overline{A}_2$, it is easy to verify $F_2 \neq -\kappa \overline{E}_2$ because of $c_1 \neq c_2$. Thus, the defined β-field does not satisfy the single-valuedness condition of displacements, and it is not a derivative field.

In the second example, the complex potentials for β-field are defined as

$$\phi_{(\beta)}(z) = \mathrm{i}\phi(z) = \mathrm{i}\left[A_1 z + A_2 \log z + a_0 + \sum_{k=1}^{\infty} \frac{a_k}{z^k}\right] \tag{24}$$

$$\psi_{(\beta)}(z) = -\mathrm{i}\psi(z) = -\mathrm{i}\left[B_1 z + B_2 \log z + b_0 + \sum_{k=1}^{\infty} \frac{b_k}{z^k}\right] \tag{25}$$

Comparing Eqs. (9) and (10) with Eqs. (24) and (25), it follows

$$E_2 = \mathrm{i}A_2, \quad F_2 = -\mathrm{i}B_2 \tag{26}$$

Since $B_2 = -\kappa \overline{A}_2$, it is easy to see that the condition $F_2 = -\kappa \overline{E}_2$ is also satisfied. Thus, the defined β-field satisfy the single-valuedness condition of displacements, and it is a derivative field.

4. Analysis of the L-integral in plane elasticity

In the literature, the L-integral is defined by

$$L(\Lambda) = \int_{(x_0,y_0),(\Lambda)}^{(x,y)} e_{3ij}[W x_j n_i + T_i u_j - T_k u_{k,i} x_j]\,\mathrm{d}s \tag{27}$$

where "Λ" denotes an arbitrary path leading from the point (x_0, y_0) to the point (x, y) (Fig. 1), e_{3ij} is the alternating tensor with the definition $(e_{312} = 1, e_{321} = -1, e_{311} = e_{322} = 0)$, u_j is the displacement, $u_{k,i} = \partial u_k/\partial x_i$, n_i is the direction cosine, $T_k = \sigma_{km} n_m$ denotes the force component, and W $(= \sigma_{ij}\varepsilon_{ij}/2)$ the strain energy function. Clearly, the integral $L(\Lambda)$ is path independent.

The integral may be rewritten in the form

$$L(\Lambda) = \int_{(x_0,y_0),(\Lambda)}^{(x,y)} [e_{3ij} W x_j n_i - e_{3ij}(-\delta_{ki} u_j + u_{k,i} x_j)\sigma_{km} n_m]\,\mathrm{d}s \tag{28}$$

where Kronecker deltas δ_{ij} is defined as $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$. In addition, we can define the integral

$$L(\mathrm{CH}) = \oint_{\mathrm{CH}} [e_{3ij} W x_j n_i - e_{3ij}(-\delta_{ki} u_j + u_{k,i} x_j)\sigma_{km} n_m]\,\mathrm{d}s \tag{29}$$

where "CH" is a closed integration path indicated in Fig. 1. Eq. (29) reveals that the terms $e_{3ij}(-\delta_{ki} u_j + u_{k,i} x_j)$ $(k = 1, 2)$ may be a displacement of a real elastic solution, which is named β-field here. Therefore, we can assume

$$u_{(\beta)} = -v + y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y}, \quad v_{(\beta)} = u + y\frac{\partial v}{\partial x} - x\frac{\partial v}{\partial y} \tag{30}$$

It is easy to verify that the displacements $u_{(\beta)}$ and $v_{(\beta)}$ satisfy the following governing equation for displacements:

$$(\kappa+1)\frac{\partial^2 u_{(\beta)}}{\partial x^2}+(\kappa-1)\frac{\partial^2 u_{(\beta)}}{\partial y^2}+2\frac{\partial^2 v_{(\beta)}}{\partial x\partial y}=0,\quad (\kappa+1)\frac{\partial^2 v_{(\beta)}}{\partial y^2}+(\kappa-1)\frac{\partial^2 v_{(\beta)}}{\partial x^2}+2\frac{\partial^2 u_{(\beta)}}{\partial x\partial y}=0 \tag{31}$$

Alternatively speaking, the displacements $u_{(\beta)}$ and $v_{(\beta)}$ is an elasticity solution. Meantime, the relevant stress components can be expressed as

$$\sigma_{x(\beta)}=-2\sigma_{xy}+y\frac{\partial\sigma_x}{\partial x}-x\frac{\partial\sigma_x}{\partial y},\quad \sigma_{y(\beta)}=2\sigma_{xy}+y\frac{\partial\sigma_y}{\partial x}-x\frac{\partial\sigma_y}{\partial y},\quad \sigma_{xy(\beta)}=\sigma_x-\sigma_y+x\frac{\partial\sigma_x}{\partial x}-y\frac{\partial\sigma_y}{\partial y} \tag{32}$$

On the other hand, we can prove that the relevant complex potentials for the β-field are as follows:

$$\phi_{(\beta)}(z)=-\mathrm{i}(z\phi'(z)-\phi(z)) \tag{33}$$

$$\psi_{(\beta)}(z)=-\mathrm{i}(z\psi'(z)+\psi(z)) \tag{34}$$

In addition, we can prove that, if the complex potentials $\phi_{(\beta)}(z)$ and $\psi_{(\beta)}(z)$ are expanded in the form of Eqs. (9) and (10), the condition (11) is satisfied. Therefore, the mentioned β-stress field is a derivative stress field. Alternatively, from Eq. (29), $L(\mathrm{CH})$ may be written in the form

$$L(\mathrm{CH})=\oint_{\mathrm{CH}}[e_{3ij}Wx_jn_i-u_{i(\beta)}\sigma_{ij}n_j]\,\mathrm{d}s \tag{35}$$

In addition, we can introduce an MWDI as follows:

$$Q(\mathrm{CH})=\frac{1}{2}\oint_{\mathrm{CH}}\{u_i\sigma_{ij(\beta)}-u_{i(\beta)}\sigma_{ij}\}n_j\,\mathrm{d}s \tag{36}$$

A relation between $L(\mathrm{CH})$ and $Q(\mathrm{CH})$ has been found, and it reads

$$L(\mathrm{CH})=Q(\mathrm{CH}) \tag{37}$$

In fact, from Eqs. (35) and (36) we can see that in order to prove Eq. (37) it is necessary to prove the following equality:

$$2\oint_{\mathrm{CH}}e_{3ij}Wx_jn_i\,\mathrm{d}s=\oint_{\mathrm{CH}}\{u_i\sigma_{ij(\beta)}+u_{i(\beta)}\sigma_{ij}\}n_j\,\mathrm{d}s \tag{38}$$

The equality (38) can be proved by considering the following points (Chen, 1985; Chen and Lee, 2002):

(a) The strain energy function can be expressed in an explicit form

$$W=\sigma_{ij}u_{i,j}/2 \tag{39}$$

(b) Since the equilibrium condition is satisfied, the following substitutions are used:

$$\frac{\partial\sigma_x}{\partial x}=-\frac{\partial\sigma_{xy}}{\partial y},\quad \frac{\partial\sigma_{xy}}{\partial x}=-\frac{\partial\sigma_y}{\partial y} \tag{40}$$

(c) In the derivation it is assumed that the single-valuedness condition of displacements is satisfied.

Since the closed path "CR" is a particular type "CH", from Eq. (37) it follows (Fig. 1)

$$L(\mathrm{CR})=Q(\mathrm{CR}) \tag{41}$$

where

$$L(\mathrm{CR})=\oint_{\mathrm{CR}}\{e_{3ij}Wx_jn_i-u_{i(\beta)}\sigma_{ij}n_j\}\,\mathrm{d}s \tag{42}$$

$$Q(\mathrm{CR}) = \frac{1}{2}\oint_{\mathrm{CR}} \{u_i\sigma_{ij(\beta)} - u_{i(\beta)}\sigma_{ij}\} n_j \, \mathrm{d}s \tag{43}$$

Note that the integral $Q(\mathrm{CR})$ is exactly the same kind of $D(\mathrm{CR})$, which was defined by Eq. (12).

Since the β-field is a derivative stress field, substituting Eqs. (4) and (5) into Eqs. (33) and (34) yields

$$\phi_{(\beta)}(z) = -\mathrm{i}(z\phi'(z) - \phi(z)) = -\mathrm{i}\left[\left(A_1 z + A_2 - \sum_{k=1}^{\infty}\frac{ka_k}{z^k}\right) - \left(A_1 z + A_2 \log z + a_0 + \sum_{k=1}^{\infty}\frac{a_k}{z^k}\right)\right] \tag{44}$$

$$\psi_{(\beta)}(z) = -\mathrm{i}(z\psi'(z) + \psi(z)) = -\mathrm{i}\left[\left(B_1 z + B_2 - \sum_{k=1}^{\infty}\frac{kb_k}{z^k}\right) + \left(B_1 z + B_2 \log z + b_0 + \sum_{k=1}^{\infty}\frac{b_k}{z^k}\right)\right] \tag{45}$$

Comparing Eqs. (44) and (45) with Eqs. (9) and (10), it is found

$$E_1 = 0, \quad E_2 = \mathrm{i}A_2, \quad e_0 = -\mathrm{i}(A_2 - a_0), \quad e_1 = \mathrm{i}(2a_1) \tag{46}$$

$$F_1 = -\mathrm{i}(2B_1), \quad F_2 = -\mathrm{i}B_2, \quad f_0 = -\mathrm{i}(B_2 + b_0), \quad f_1 = 0 \tag{47}$$

The constants $A_1, A_2, a_0, a_1, \ldots B_1, B_2, b_0, b_1, \ldots$ for the α-field were shown in Eqs. (4), (5), (7) and (8). The constants $E_1, E_2, e_0, e_1, \ldots F_1, F_2, f_0, f_1, \ldots$ for the β-field were shown in Eqs. (46) and (47). Substituting the mentioned constants into Eq. (20) yields

$$L(\mathrm{CR}) = Q(\mathrm{CR}) = \frac{\pi(\kappa+1)}{G}\mathrm{Im}[A_2 b_0 - a_0 B_2 - 2a_1 B_1] \tag{48}$$

Some particular features can be found from Eq. (48): (a) the $L(\mathrm{CR})$ value does not depend on A_1 and b_1, (b) for evaluating the $L(\mathrm{CR})$, it is necessary to obtain the constants A_2, a_0, a_1 and B_1, B_2, b_0.

For a given stress state of the infinite plate, consider how the $L(\mathrm{CR})$-integral changes from one system of rectangular coordinates to another. Let (x, y) and (x_*, y_*) be the coordinates of the same point in the (xoy) and $(x_* o_* y_*)$ systems and let

$$z_* = z + z_d \quad (\text{with } z_* = x_* + \mathrm{i}y_*,\ z = x + \mathrm{i}y,\ z_d = x_d + \mathrm{i}y_d) \tag{49}$$

where z_d represents a translation of the coordinate system (Fig. 2(a)).

The complex potentials in the new coordinates (x_*, y_*) take the form (Muskhelishvili, 1953)

$$\phi_*(z_*) = \phi(z_* - z_d), \quad \psi_*(z_*) = \psi(z_* - z_d) - \bar{z}_d\phi'(z_* - z_d) \tag{50}$$

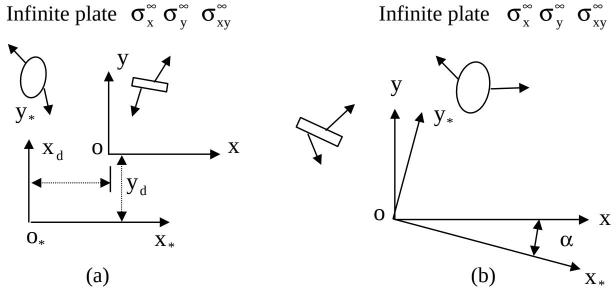

Fig. 2. Transformation of the coordinates: (a) translation of the coordinates, (b) rotation of the coordinates.

The complex potentials in the $(x_* o_* y_*)$ coordinates can be expressed in the following form:

$$\phi_*(z_*) = A_{1*}z_* + A_{2*}\log z_* + a_{0*} + \sum_{k=1}^{\infty} \frac{a_{k*}}{z_*^k} \tag{51}$$

$$\psi_*(z_*) = B_{1*}z_* + B_{2*}\log z_* + b_{0*} + \sum_{k=1}^{\infty} \frac{b_{k*}}{z_*^k} \tag{52}$$

After substituting Eqs. (4) and (5) into Eq. (50), it is easy to find the following results:

$$A_{1*} = A_1, \quad A_{2*} = A_2, \quad a_{0*} = a_0 - A_1 z_d, \quad a_{1*} = a_1 - A_2 z_d \tag{53}$$

$$B_{1*} = B_1, \quad B_{2*} = B_2, \quad b_{0*} = b_0 - B_1 z_d - A_1 \bar{z}_d, \quad b_{1*} = b_1 - B_2 z_d - A_2 \bar{z}_d \tag{54}$$

In the $(x_* o_* y_*)$ coordinates, the $L_*(\mathrm{CR})$ can be expressed in the form

$$L_*(\mathrm{CR}) = \frac{\pi(\kappa+1)}{G} \mathrm{Im}[A_{2*}b_{0*} - a_{0*}B_{2*} - 2a_{1*}B_{1*}] \tag{55}$$

Substituting Eqs. (53) and (54) into Eq. (55) yields

$$L_*(\mathrm{CR}) = L(\mathrm{CR}) + L_a \tag{56}$$

where

$$L_a = y_d J_1(\mathrm{CR}) - x_d J_2(\mathrm{CR}) \quad (z_d = x_d + \mathrm{i}y_d) \tag{57}$$

$$J_1(\mathrm{CR}) = \oint_{\mathrm{CR}} \left[W n_1 - \frac{\partial u_i}{\partial x}\sigma_{ij}n_j \right] \mathrm{d}s = \frac{\pi(\kappa+1)}{G} \mathrm{Re}[A_1 A_2 + A_1 B_2 + A_2 B_1] \tag{58}$$

$$J_2(\mathrm{CR}) = \oint_{\mathrm{CR}} \left[W n_2 - \frac{\partial u_i}{\partial y}\sigma_{ij}n_j \right] \mathrm{d}s = \frac{\pi(\kappa+1)}{G} \mathrm{Im}[A_1 A_2 - A_1 B_2 - A_2 B_1] \tag{59}$$

In Eqs. (57)–(59), the J_1 and J_2 are well known in the literature (Chen and Lee, 2002). Eq. (55) reveals that the $L(\mathrm{CR})$ is not an invariant with respect to the translation of coordinates.

Secondly, consider how the L-integral changes under a rotational transformation of the rectangular coordinates (Fig. 2(b)). Let (x, y) and (x_*, y_*) be the coordinates of the same point in the (xoy) and $(x_* o y_*)$ systems and let

$$z_* = z\exp(\mathrm{i}\alpha) \quad (\text{with } z_* = x_* + \mathrm{i}y_*,\ z = x + \mathrm{i}y) \tag{60}$$

where the angle α represents a rotation of the coordinate system (Fig. 2(b)).

The complex potentials in the new coordinates (x_*, y_*) take the form (Muskhelishvili, 1953)

$$\phi_*(z_*) = \exp(\mathrm{i}\alpha)\phi[\exp(-\mathrm{i}\alpha)z_*], \quad \psi_*(z_*) = \exp(-\mathrm{i}\alpha)\psi[\exp(-\mathrm{i}\alpha)z_*] \tag{61}$$

Similarly, the complex potentials can be expressed in the following forms:

$$\phi_*(z_*) = A_{1*}z_* + A_{2*}\log z_* + a_{0*} + \sum_{k=1}^{\infty} \frac{a_{k*}}{z_*^k} \tag{62}$$

$$\psi_*(z_*) = B_{1*}z_* + B_{2*}\log z_* + b_{0*} + \sum_{k=1}^{\infty} \frac{b_{k*}}{z_*^k} \tag{63}$$

After substituting Eqs. (4) and (5) into Eq. (61), it is easy to find the following results:

$$A_{1*} = A_1, \quad A_{2*} = \exp(\mathrm{i}\alpha)A_2, \quad a_{0*} = \exp(\mathrm{i}\alpha)[-\mathrm{i}\alpha A_2 + a_0], \quad a_{1*} = \exp(2\mathrm{i}\alpha)a_1 \tag{64}$$

$$B_{1*} = \exp(-2i\alpha)B_1, \quad B_{2*} = \exp(-i\alpha)B_2, \quad b_{0*} = \exp(-i\alpha)[-i\alpha B_2 + b_0], \quad b_{1*} = b_1 \tag{65}$$

Similarly, in the $(x_* o_* y_*)$ coordinates the $L_*(\mathrm{CR})$ can be expressed in the form

$$L_*(\mathrm{CR}) = \frac{\pi(\kappa+1)}{G}\mathrm{Im}[A_{2*}b_{0*} - a_{0*}B_{2*} - 2a_{1*}B_{1*}] \tag{66}$$

Substituting Eqs. (64) and (65) into Eq. (66) yields

$$L_*(\mathrm{CR}) = L(\mathrm{CR}), \tag{67}$$

Eq. (67) reveals that the $L(\mathrm{CR})$ is an invariant with respect to the rotation of coordinates.

Three particular cases for evaluating the L-integral are introduced below.

(1) In the first case, the loading condition is shown by Fig. 3(a) where the concentrated forces P_x and P_y are applied on the crack face, the available solution is as follows (Chen, 1995):

$$\begin{matrix} \phi'(z) \\ \omega'(z) \end{matrix} = \frac{P_x + iP_y}{2\pi}\left[\frac{\kappa-1}{\kappa+1}\frac{1}{X(z)} \mp \frac{1}{z-s}\right] \tag{68}$$

$$\psi'(z) = \bar{\omega}'(z) - z\phi''(z) - \phi'(z) \tag{69}$$

where

$$X(z) = \sqrt{z^2 - a^2} \quad (\text{taking the branch } \mathrm{Lim}_{z\to\infty}X(z)/z = 1) \tag{70}$$

After comparing Eqs. (68) and (69) with Eqs. (4) and (5), we will find

$$A_1 = 0, \quad A_2 = -\frac{P_x + iP_y}{\pi(\kappa+1)}, \quad a_1 = \frac{(P_x + iP_y)s}{2\pi} \tag{71}$$

$$B_1 = 0, \quad B_2 = \frac{\kappa(P_x - iP_y)}{\pi(\kappa+1)}, \quad b_1 = \frac{(2iP_y)s}{2\pi} \tag{72}$$

Using Eqs. (48), (71) and (72) yields

$$L(\mathrm{CR}) = -\frac{1}{G}\mathrm{Im}[(\kappa a_0 - \bar{b}_0)(P_x - iP_y)] \tag{73}$$

It is known that for a given stress state, two constants a_0, b_0 in Eqs. (4) and (5) cannot be determined from the force boundary condition. That is to say in the present case the $L(\mathrm{CR})$ value may be influenced by the rigid motion of body, which is determined by a_0 and b_0.

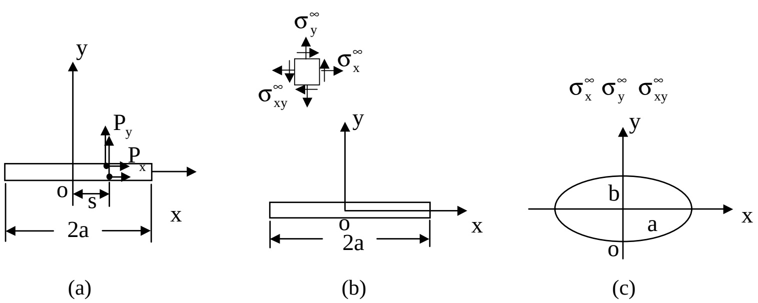

Fig. 3. Three loading cases: (a) a crack with forces on the crack face, (b) a crack with remote loading, (c) an elliptic hole in an infinite plate.

(2) In the second case, the loading condition is shown by Fig. 3(b) where the remote tractions σ_x^∞, σ_y^∞ and σ_{xy}^∞ are applied, the available solution is as follows:

$$\phi'(z) = \frac{1}{4}(\sigma_x^\infty + \sigma_y^\infty) - \frac{1}{2}(\sigma_y^\infty - \mathrm{i}\sigma_{xy}^\infty)\left(1 - \frac{z}{X(z)}\right) \tag{74}$$

$$\psi'(z) = \frac{1}{2}(\sigma_y^\infty - \sigma_x^\infty + 2\mathrm{i}\sigma_{xy}^\infty) - \mathrm{i}\sigma_{xy}^\infty\left(1 - \frac{z}{X(z)}\right) + \frac{1}{2}(\sigma_y^\infty - \mathrm{i}\sigma_{xy}^\infty)\frac{a^2 z}{X^3(z)} \tag{75}$$

After comparing Eqs. (74) and (75) with Eqs. (4) and (5), we will find

$$A_1 = \frac{\sigma_x^\infty + \sigma_y^\infty}{4}, \quad A_2 = 0, \quad a_1 = -\frac{1}{4}(\sigma_y^\infty - \mathrm{i}\sigma_{xy}^\infty)a^2 \tag{76}$$

$$B_1 = \frac{1}{2}(\sigma_y^\infty - \sigma_x^\infty + 2\mathrm{i}\sigma_{xy}^\infty), \quad B_2 = 0, \quad b_1 = -\frac{1}{2}\sigma_y^\infty a^2 \tag{77}$$

Using Eqs. (48), (76) and (77) yields (Herrmann and Herrmann, 1981)

$$L(\mathrm{CR}) = \frac{\pi(\kappa+1)}{4G}\sigma_{xy}^\infty(\sigma_x^\infty + \sigma_y^\infty)a^2 \tag{78}$$

or

$$L(\mathrm{CR}) = \frac{\kappa+1}{4G}K_2(K_1 + \sigma_x^\infty\sqrt{\pi a})a \tag{79}$$

where K_1 and K_2 are the stress intensity factor at the crack tip.

(3) In the third example, it is assumed that an infinite plate containing an elliptical notch is applied by the remote tractions σ_x^∞, σ_y^∞ and σ_{xy}^∞ and that the notch surface is traction free (Fig. 3(c)). In the case, the previously introduced complex potentials $\phi(z)$ and $\psi(z)$ are rewritten as $\phi_1(z)$ and $\psi_1(z)$. Meantime, the following mapping function is introduced:

$$z = \omega(\varsigma) = R_0\left(\varsigma + \frac{m}{\varsigma}\right) \quad (0 \leqslant m \leqslant 1) \tag{80}$$

which maps the region outside the unit circle (in ς-plane) into a region outside the elliptical notch (in z-plane). Clearly, there are the following relations:

$$a = R_0(1+m), \quad b = R_0(1-m), \quad R_0 = (a+b)/2, \quad m = (a-b)/(a+b) \tag{81}$$

where "a" and "b" are the major and minor axes of the elliptical notch, respectively (Fig. 3(c)). In ς-plane two complex potentials are defined

$$\phi(\varsigma) = \phi_1(z)\Big|_{z=\omega(\varsigma)}, \quad \psi(\varsigma) = \psi_1(z)\Big|_{z=\omega(\varsigma)} \tag{82}$$

The traction free condition of the elliptical contour leads to

$$\phi(\varsigma) + \omega(\varsigma)\frac{\overline{\phi'(\varsigma)}}{\overline{\omega'(\varsigma)}} + \overline{\psi(\varsigma)} = 0 \quad (\text{on unit circle } |\varsigma| = 1) \tag{83}$$

After some manipulation, the boundary value problem has the following solution (Muskhelishvili, 1953):

$$\phi(\varsigma) = A_1 R_0 \varsigma - (mA_1 + \overline{B}_1)R_0\frac{1}{\varsigma} \tag{84}$$

$$\psi(\varsigma) = B_1R_0\varsigma - A_1R_0\frac{1}{\varsigma} - (1+m^2)A_1R_0\frac{\varsigma}{\varsigma^2 - m} - (mA_1 + \overline{B}_1)R_0\frac{m\varsigma^2+1}{\varsigma(\varsigma^2 - m)} \tag{85}$$

where

$$A_1 = \frac{\sigma_x^\infty + \sigma_y^\infty}{4}, \quad B_1 = \frac{\sigma_y^\infty - \sigma_x^\infty}{2} + \mathrm{i}\sigma_{xy}^\infty \tag{86}$$

From Eq. (80), at the remote place we have

$$\varsigma \approx \frac{z}{R_0} - \frac{mR_0}{z}, \quad \frac{1}{\varsigma} \approx \frac{R_0}{z} \tag{87}$$

Substituting Eq. (87) into Eqs. (84) and (85) yields

$$\phi_1(z) = \phi(\varsigma)_{\varsigma=\omega^{-1}(z)} = A_1z + A_2\log z + \frac{a_1}{z} + \cdots \tag{88}$$

$$\psi_1(z) = \psi(\varsigma)_{\varsigma=\omega^{-1}(z)} = B_1z + B_2\log z + \frac{b_1}{z} + \cdots \tag{89}$$

where

$$A_2 = B_2 = 0, \quad a_1 = -R_0^2(2mA_1 + \overline{B}_1), \quad b_1 = -R_0^2[2(1+m^2)A_1 + mB_1 + m\overline{B}_1] \tag{90}$$

Substituting Eqs. (86) and (90) into Eq. (48) yields

$$L(\mathrm{CR}) = \frac{\pi(\kappa+1)}{G}mR_0^2\sigma_{xy}^\infty(\sigma_x^\infty + \sigma_y^\infty) \tag{91}$$

5. Conclusions

The derivative stress field plays an important role in the present study. It is found that, for the cases of J_1-, J_2-, L-, and M-integral, the relevant derivative stress fields can be found from this study and previous papers (Chen, 1985; Chen and Lee, 2002). If the complex potentials are expressed in the form of Eqs. (4) and (5), the final results are as follows:

$$J_1(\mathrm{CR}) = \oint_{\mathrm{CR}}\left[Wn_1 - \frac{\partial u_i}{\partial x}\sigma_{ij}n_j\right]\mathrm{d}s = \frac{\pi(\kappa+1)}{G}\mathrm{Re}[A_1A_2 + A_1B_2 + A_2B_1] \tag{92}$$

$$J_2(\mathrm{CR}) = \oint_{\mathrm{CR}}\left[Wn_2 - \frac{\partial u_i}{\partial y}\sigma_{ij}n_j\right]\mathrm{d}s = \frac{\pi(\kappa+1)}{G}\mathrm{Im}[A_1A_2 - A_1B_2 - A_2B_1] \tag{93}$$

$$L(\mathrm{CR}) = \oint_{\mathrm{CR}}[e_{3ij}Wx_jn_i - e_{3ij}(-\delta_{ki}u_j + u_{k,i}x_j)\sigma_{km}n_m]\,\mathrm{d}s = \frac{\pi(\kappa+1)}{G}\mathrm{Im}[A_2b_0 - a_0B_2 - 2a_1B_1] \tag{94}$$

$$M(\mathrm{CR}) = \oint_{\mathrm{CR}}(Wx_in_i - u_{i,k}x_k\sigma_{ij}n_j)\,\mathrm{d}s = \frac{\pi(\kappa+1)}{G}\mathrm{Re}[-A_1b_1 - a_1B_1 + A_2B_2] \tag{95}$$

Previously, researchers were unaware of the vanishing condition for the mentioned path independent integrals. For the J_1(CR)- and J_2(CR)-integrals, two sufficient conditions for the vanishing value of the integrals are as follows: (a) $A_1 = B_1 = 0$ (or $\sigma_x^\infty = \sigma_y^\infty = \sigma_{xy}^\infty = 0$) or (b) $A_2 = B_2 = 0$ (or $F_x = F_y = 0$).

Meantime, for the L(CR)-integral, a sufficient condition for the vanishing value of the integral is as follows: $B_1 = 0$ (or $\sigma_y^\infty - \sigma_x^\infty + 2i\sigma_{xy}^\infty = 0$) and $A_2 = B_2 = 0$ (or $F_x = F_y = 0$). Finally, for the M(CR)-integral, a sufficient condition for the vanishing value of the integral is as follows: (a) $A_1 = B_1 = 0$ (or $\sigma_x^\infty = \sigma_y^\infty = \sigma_{xy}^\infty = 0$) and (b) $A_2 = B_2 = 0$ (or $F_x = F_y = 0$). From the above-mentioned analysis we see that the vanishing conditions for these integrals are quite different.

Obviously, it is straightforward to obtain the path independent integrals (J_1(CR)-, J_2(CR)-, L(CR)- and M(CR)-integrals) from the general results shown by Eqs. (92)–(95). However, it is not easy to get the physical explanation for those integrals in more general case, particularly in the case of $A_2 \neq 0$ and $B_2 \neq 0$, i.e. there are some resultant forces applied on the finite portion of infinite plate.

These integrals can be evaluated for a simple crack with the conditions: (a) there is no resultant forces applied on the crack, i.e. $A_2 = B_2 = 0$, (b) the remote tractions are σ_x^∞, σ_y^∞ and σ_{xy}^∞. In this case, simply using Eqs. (7), (8), (76), (77) and (92)–(95) yields

$$J_1(\mathrm{CR}) = \oint_{\mathrm{CR}} \left[W n_1 - \frac{\partial u_i}{\partial x} \sigma_{ij} n_j \right] \mathrm{d}s = 0 \tag{96}$$

$$J_2(\mathrm{CR}) = \oint_{\mathrm{CR}} \left[W n_2 - \frac{\partial u_i}{\partial y} \sigma_{ij} n_j \right] \mathrm{d}s = 0 \tag{97}$$

$$L(\mathrm{CR}) = \oint_{\mathrm{CR}} [e_{3ij} W x_j n_i - e_{3ij}(-\delta_{ki} u_j + u_{k,i} x_j) \sigma_{km} n_m] \, \mathrm{d}s = \frac{\pi(\kappa+1)}{4G} \sigma_{xy}^\infty (\sigma_x^\infty + \sigma_y^\infty) a^2 \tag{98}$$

$$M(\mathrm{CR}) = \oint_{\mathrm{CR}} (W x_i n_i - u_{i,k} x_k \sigma_{ij} n_j) \, \mathrm{d}s = \frac{\pi(\kappa+1)}{4G} [(\sigma_y^\infty)^2 + (\sigma_{xy}^\infty)^2] a^2 \tag{99}$$

Clearly, M(CR) can serve as a measure of severity at the crack tip. If the values of σ_y^∞ and σ_{xy}^∞ are higher, the crack tip is more dangerous. However, one cannot see the role of the component σ_x^∞ in the expression of M(CR). On the other hand, the L(CR) may be another measure of severity at the crack tip, in which all the components σ_x^∞, σ_y^∞ and σ_{xy}^∞ are involved. However, unlike the M(CR) case, one cannot judge whether the L(CR) is definitely positive or negative.

It is known that the M(CR) is an additional strain energy stored in a cracked body. Therefore, it is interesting to investigate the M(CR) variation with respect to a neighboring configuration of the crack (Fig. 4).

In the first case, we assume that the crack length "2a" get an extension and becomes $2a(1+\alpha)$ (Fig. 4(b)). In this case, the relevant M(CR) is denoted by

$$M(\alpha) = \frac{\pi(\kappa+1)}{4G} [(\sigma_y^\infty)^2 + (\sigma_{xy}^\infty)^2](1+\alpha)^2 a^2 \tag{100}$$

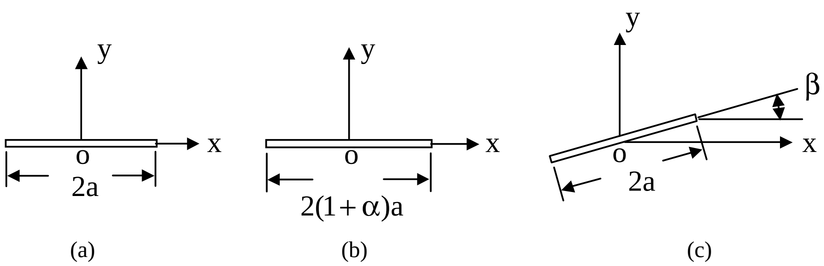

Fig. 4. A single crack with remote tractions σ_x^∞, σ_y^∞ and σ_{xy}^∞: (a) the original case, (b) a crack with extension, (c) a crack with rotation.

Clearly, from Eq. (100) we have

$$\left.\frac{\partial M(\alpha)}{\partial \alpha}\right|_{\alpha=0} = 2M(\mathrm{CR}) \tag{101}$$

That is to say, the value of $M(\mathrm{CR})$ represents the energy variation with respect to the crack extension.

In the second case, we assume that the crack get a rotation with a rotating angle "β" (Fig. 4(c)). In this case, the relevant $M(\mathrm{CR})$ is denoted by

$$M(\beta) = \frac{\pi(\kappa+1)}{4G}[(\sigma_{y*}^{\infty})^2 + (\sigma_{xy*}^{\infty})^2]a^2 \tag{102}$$

where

$$\begin{aligned} \sigma_{y*}^{\infty} &= \sigma_x^{\infty}\sin^2\beta + \sigma_y^{\infty}\cos^2\beta - 2\sigma_{xy}^{\infty}\sin\beta\cos\beta \\ \sigma_{xy*}^{\infty} &= (-\sigma_x^{\infty} + \sigma_y^{\infty})\sin\beta\cos\beta + \sigma_{xy}^{\infty}(\cos^2\beta - \sin^2\beta) \end{aligned} \tag{103}$$

Clearly, from Eq. (102) we have

$$\left.\frac{\partial M(\beta)}{\partial \beta}\right|_{\beta=0} = -2L(\mathrm{CR}) \tag{104}$$

That is to say, the value of $L(\mathrm{CR})$ represents the energy variation with respect to the crack rotation. Similar derivation can be found from Herrmann and Herrmann (1981).

Clearly, two equations (101) and (104) are similar. However, the former case shown by Fig. 4(b) can be realized physically. For example, in the case of $\sigma_y^{\infty} \neq 0$ and $\sigma_{xy}^{\infty} = 0$, the crack may have extension in x-direction. However, the case of crack rotation does not exist physically (Fig. 4(c)).

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